Factorization of the Non-Stationary Schrödinger Operator

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Abstract

We consider a factorization of the non-stationary Schrödinger operator based on the parabolic Dirac operator introduced by Cerejeiras/ Kähler/ Sommen. Based on the fundamental solution for the parabolic Dirac operators, we shall construct appropriated Teodorescu and Cauchy-Bitsadze operators. Afterwards we will describe how to solve the nonlinear Schrödinger equation using Banach fixed point theorem.

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1 Introduction

Time evolution problems are of extreme importance in mathematical physics. However, there is still a need for special techniques to deal with these problems, specially when non-linearities are involved.

For stationary problems, the theory developed by K. Gürlebeck and W. Sprößig [10], based on an orthogonal decomposition of the underlying function space in terms of the subspace of null-solutions of the corresponding Dirac operator, has been successfully applied to a wide range of equations, for instance Lamé, Navier-Stokes, Maxwell or Schrödinger equations [6], [10], [11], [2] or [13]. Unfortunately, there is no easy way to extend this theory directly to non-stationary problems.

In [7] the authors proposed an alternative approach in terms of a Witt basis. This approach allowed a successful application of the already existent techniques of elliptic function theory (see [10], [6]) to non-stationary problems in time-varying domains. Namely, a suitable orthogonal decomposition for the

underlying function space was obtained in terms of the kernel of the parabolic Dirac operator and its range after application to a Sobolev space with zero boundary-values.

In this paper we wish to apply this approach to study the existence and uniqueness of solutions of the non-stationary nonlinear Schrödinger equation.

Initially, in section two, we will present some basic notions about complexified Clifford algebras and Witt basis. In section three we will present a factorization for the operators $(\pm i\partial_t - \Delta)$ using an extension of the parabolic Dirac operator introduced in [7]. For the particular case of the non-stationary Schrödinger operator we will present the corresponding Teodorescu and Cauchy-Bitsadze operators in analogy to [10]. Moreover, we will obtain some direct results about the decomposition of L_p-spaces and the resolution of the linear Schrödinger problem.

In the last section we will present an algorithm to solve numerically the non-linear Schrödinger and we prove its convergence in L_2-sense using Banach's fixed point theorem.

2 Preliminaries

We consider the m-dimensional vector space $\mathbb{R}^m$ endowed with an orthonormal basis $\{e_1, \cdots, e_m\}$.

We define the universal Clifford algebra $C\ell_{0,m}$ as the 2^m-dimensional associative algebra which preserves the multiplication rules $e_ie_j + e_je_i = -2\delta_{i,j}$. A basis for $C\ell_{0,m}$ is given by $e_0 = 1$ and $e_A = e_{h_1} \cdots e_{h_k}$, where $A = \{h_1, \ldots, h_k\} \subset M = \{1, \ldots, m\}$, for $1 \leq h_1 < \cdots < h_k \leq m$. Each element $x \in C\ell_{0,m}$ will be represented by $x = \sum_A x_A e_A$, $x_A \in \mathbb{R}$, and each non-zero vector $x = \sum_{j=1}^m x_j e_j \in \mathbb{R}^m$ has a multiplicative inverse given by $\frac{-x}{|x|^2}$. We denote by $\overline{x}^{C\ell_{0,m}}$ the (Clifford) conjugate of the element $x \in C\ell_{0,m}$, where

$$\overline{1}^{C\ell_{0,m}} = 1,\ \overline{e_j}^{C\ell_{0,m}} = -e_j,\ \overline{ab}^{C\ell_{0,m}} = \overline{b}^{C\ell_{0,m}}\overline{a}^{C\ell_{0,m}}.$$

We introduce the complexified Clifford algebra $C\ell_m$ as the tensorial product

$$\mathbb{C} \otimes C\ell_{0,m} = \left\{ w = \sum_A z_A e_A,\ z_A \in \mathbb{C}, A \subset M \right\}$$

where the imaginary unit interacts with the basis elements via $ie_j = e_j i, j = 1, \ldots, m$. The conjugation in $C\ell_m = \mathbb{C} \otimes C\ell_{0,m}$ will be defined as $\overline{w} = \sum_A \overline{z_A}^{\mathbb{C}} \overline{e_A}^{C\ell_{0,m}}$. Let us remark that for $a, b \in C\ell_m$ we have $|ab| \leq 2^m |a||b|$.

We introduce the Dirac operator $D = \sum_{j=1}^m e_j \partial_{x_i}$. It factorizes the m-dimensional Laplacian, that is, $D^2 = -\Delta$. A $C\ell_m$-valued function defined on an open domain $\underline{\Omega}$, $u : \underline{\Omega} \subset \mathbb{R}^m \mapsto C\ell_m$, is said to be *left-monogenic* if it satisfies $Du = 0$ on $\underline{\Omega}$ (resp. *right-monogenic* if it satisfies $uD = 0$ on $\underline{\Omega}$).

A function $u : \underline{\Omega} \mapsto C\ell_m$ has a representation $u = \sum_A u_A e_A$ with $\mathbb{C}$-valued components u_A. Properties such as continuity will be understood componentwisely. In the following we will use the short notation $L_p(\underline{\Omega})$, $C^k(\underline{\Omega})$, etc.,

instead of $L_p(\underline{\Omega}, C\ell_m)$, $C^k(\underline{\Omega}, C\ell_m)$. For more details on Clifford analysis, see [5], [12], [4] or [9].

Taking into account [7] we will imbed $\mathbb{R}^m$ into $\mathbb{R}^{m+2}$. For that purpose we add two new basis elements $\mathfrak{f}$ and $\mathfrak{f}^\dagger$ satisfying

$$\mathfrak{f}^2 = \mathfrak{f}^{\dagger 2} = 0, \quad \mathfrak{f}\mathfrak{f}^\dagger + \mathfrak{f}^\dagger\mathfrak{f} = 1, \quad \mathfrak{f}e_j + e_j\mathfrak{f} = \mathfrak{f}^\dagger e_j + e_j\mathfrak{f}^\dagger = 0, j = 1, \cdots, m.$$

This construction will allows us to use a suitable factorization of the time evolution operators where only partial derivatives are used.

3 Factorization of time-evolution operators

In this section we will study the forward/backward Schrödinger equations,

$$(\pm i\partial_t - \Delta)u(x,t) = 0, \quad (x,t) \in \Omega, \tag{1}$$

where $\Omega \subset \mathbb{R}^m \times \mathbb{R}^+$, $m \geq 3$, stands for an open domain in $\mathbb{R}^m \times \mathbb{R}^+$. We remark at this point that Ω is a time-variating domain and, therefore, not necessarily a cylindric domain.

Taking account the ideas presented in [1] and [7] we introduce the following definition

Definition 3.1. *For a function $u \in W_p^1(\Omega)$, $1 < p < +\infty$, we define the forward (resp. backward) parabolic Dirac operator*

$$D_{x,\pm it}u = (D + \mathfrak{f}\partial_t \pm i\mathfrak{f}^\dagger)u, \tag{2}$$

where D stands for the (spatial) Dirac operator.

It is obvious that $D_{x,\pm it} : W_p^1(\Omega) \to L_p(\Omega)$.

These operators factorize the correspondent time-evolution operator (1), that is

$$(D_{x,\pm it})^2 u = (\pm i\partial_t - \Delta)u. \tag{3}$$

Moreover, we consider the generic Stokes' Theorem

Theorem 3.2. *For each $u, v \in W_p^1(\Omega)$, $1 < p < \infty$, it holds*

$$\int_\Omega v d\sigma_{x,t} u = \int_{\partial\Omega} [(vD_{x,-it})u + v(D_{x,+it}u)]dxdt$$

where the surface element is $d\sigma_{x,t} = (D_x + \mathfrak{f}\partial_t)\rfloor dxdt$, the contraction of the homogeneous operator associated to $D_{x,-it}$ with the volume element.

We now construct the fundamental solution for the time-evolution operator $-\Delta - i\partial_t$. For that purpose, we consider the fundamental solution of the heat operator

$$e(x,t) = \frac{H(t)}{(4\pi t)^{\frac{m}{2}}} \exp\left(-\frac{|x|^2}{4t}\right), \tag{4}$$

where $H(t)$ denotes the Heaviside-function. Let us remark that the previous fundamental solution verifies

$$(-\Delta + \partial_t)e(x,t) \quad = \quad \delta(x)\delta(t).$$

We apply to (4) the rotation $t \to it$. There we obtain

$$(-\Delta - i\partial_t)e(x,it) = -\Delta e(x,it) + \partial_{it} e(x,it) = \delta(x)\delta(it) = -i\delta(x)\delta(t),$$

i.e., the fundamental solution for the Schrödinger operator $-\Delta - i\partial_t$ is

$$\begin{aligned} e_-(x,t) &= ie(x,it) \\ &= i\,\frac{H(t)}{(4\pi it)^{\frac{m}{2}}}\exp\left(i\,\frac{|x|^2}{4t}\right). \end{aligned} \qquad (5)$$

Then we have

Definition 3.3. *Given the fundamental solution $e_- = e_-(x,t)$ we have as fundamental solution $E_- = E_-(x,t)$ for the parabolic Dirac operator $D_{x,-it}$ the function*

$$\begin{aligned} E_-(x,t) &= e_-(x,t)D_{x,-it} \\ &= \frac{H(t)}{(4\pi it)^{\frac{m}{2}}}\exp\left(\frac{i|x|^2}{4t}\right)\left(\frac{-x}{2t} + \mathfrak{f}\left(\frac{|x|^2}{4t^2} - \frac{im}{2t}\right) + \mathfrak{f}^\dagger\right) \end{aligned} \qquad (6)$$

If we replace the function v by the fundamental solution E_- in the generic Stoke's formula presented before, we have, for a function $u \in W_p^1(\Omega)$ and a point $(x_0,t_0) \notin \partial\Omega$, the Borel-Pompeiu formula,

$$\begin{aligned} &\int_{\partial\Omega} E_-(x-x_0,t-t_0)d\sigma_{x,t}\,u(x,t) \\ &= u(x_0,t_0) + \int_{\Omega} E_-(x-x_0,t-t_0)(D_{x,+it}u)dxdt. \end{aligned} \qquad (7)$$

Moreover, if $u \in ker(D_{x,+it})$ we obtain the Cauchy's integral formula

$$\int_{\partial\Omega} E_-(x-x_0,t-t_0)d\sigma_{x,t}\;u(x,t) = u(x_0,t_0).$$

Based on expression (7) we define the Teodorescu and Cauchy-Bitsadze operators.

Definition 3.4. *For a function $u \in L_p(\Omega)$ we have*

(a) the Teodorescu operator

$$T_-u(x_0,t_0) \quad = \quad \int_{\Omega} E_-(x-x_0,t-t_0)u(x,t)dxdt \qquad (8)$$

(b) the Cauchy-Bitsadze operator

$$F_-u(x_0,t_0) \quad = \quad \int_{\partial\Omega} E_-(x-x_0,t-t_0)d\sigma_{x,t}u(x,t), \qquad (9)$$

for $(x_0,t_0) \notin \partial\Omega$.

Using the previous operators, (7) can be rewritten as

$$F_{-}u = u + T_{-}D_{x,+it}u,$$

whenever $v \in W_p^1(\Omega),\ 1 < p < \infty$.

Moreover, the Teodurescu operator is the right inverse of the parabolic Dirac operator $D_{x,-it)}$, that is,

$$\begin{aligned} D_{x,-it}Tu &= \int_{\Omega} D_{x,-it}E_{-}(x-x_0,t-t_0)u(x,t)dxdt \\ &= \int_{\Omega} \delta(x-x_0,t-t_0)u(x,t)dxdt \\ &= u(x_0,t_0), \end{aligned}$$

for all $(x_0,t_0) \in \Omega$.

In view of the previous definitions and relations, we obtain the following results, in an analogous way as in [7].

Theorem 3.5. *If $v \in W_p^{\frac{1}{2}}(\partial\Omega)$ then the trace of the operator F_{-} is*

$$\mathrm{tr}(F_{-}v) \quad = \quad \frac{1}{2}v - \frac{1}{2}S_{-}v, \tag{10}$$

where

$$S_{-}v(x_0,t_0) \quad = \quad \int_{\partial\Omega} E_{-}(x-x_0,t-t_0)d\sigma_{x,t}v(x,t)$$

is a generalization of the Hilbert transform.

Also, the operator S_{-} satisfies $S_{-}^2 = I$ and, therefore, the operators

$$\mathbf{P} = \frac{1}{2}I + \frac{1}{2}S_{-}, \quad \mathbf{Q} = \frac{1}{2}I - \frac{1}{2}S_{-}$$

are projections into the Hardy spaces.

Taking account the ideas presented in [7] an immediate application is given by the decomposition of the L_p-space.

Theorem 3.6. *The space $L_p(\Omega)$, for $1 < p \leq 2$, allows the following decomposition*

$$L_p(\Omega) = L_p(\Omega) \cap ker\left(D_{x,-it}\right) \oplus D_{x,it}\left(\overset{\circ}{W}{}_p^1(\Omega)\right),$$

and we can define the following projectors

$$\begin{aligned} P_{-}: \quad L_p(\Omega) &\to L_p(\Omega) \cap ker\left(D_{x,-it}\right) \\ Q_{-}: \quad L_p(\Omega) &\to D_{x,-it}\left(\overset{\circ}{W}{}_p^1(\Omega)\right). \end{aligned}$$

Proof. Let us denote by $(-\Delta - i\partial_t)_0^{-1}$ the solution operator of the problem

$$\begin{cases} (-\Delta - i\partial_t)u &= f \ \ in \ \ \Omega \\ \\ u &= 0 \ \ on \ \ \partial\Omega \end{cases}$$

As a first step we take a look at the intersection of the two subspaces $D_{x,-it}\left(\overset{\circ}{W}{}_p^1(\Omega)\right)$ and $L_p(\Omega) \cap \ker(D_{x,-it})$.

Consider $u \in L_p(\Omega) \cap \ker(D_{x,-it}) \cap D_{x,-it}\left(\overset{\circ}{W}{}_p^1(\Omega)\right)$. It is immediate that $D_{x,-it}u = 0$ and also, because $u \in D_{x,-it}\left(\overset{\circ}{W}{}_p^1(\Omega)\right)$, there exist a function $v \in \overset{\circ}{W}{}_p^1(\Omega)$ with $D_{x,-it}v = u$ and $(-\Delta - i\partial_t)v = 0$.

Since $(-\Delta - i\partial_t)_0^{-1}f$ is unique (see [14]) we get $v = 0$ and, consequently, $u = 0$, i. e., the intersection of this subspaces contains only the zero function. Therefore, our sum is a direct sum.

Now let us $u \in L_p(\Omega)$. Then we have

$$u_2 = D_{x,-it}(-\Delta - i\partial_t)_0^{-1}D_{x,-it}u \ \in \ D_{x,-it}\left(\overset{\circ}{W}{}_p^1(\Omega)\right).$$

Let us now apply $D_{x,-it}$ to the function $u_1 = u - u_2$. This results in

$$\begin{aligned} D_{x,-it}u_1 &= D_{x,-it}u - D_{x,-it}u_2 \\ &= D_{x,-it}u - D_{x,-it}D_{x,-it}(-\Delta - i\partial_t)_0^{-1}D_{x,-it}u \\ &= D_{x,-it}u - (-\Delta - i\partial_t)(-\Delta - i\partial_t)_0^{-1}D_{x,-it}u \\ &= D_{x,-it}u - D_{x,-it}u \\ &= 0, \end{aligned}$$

i.e., $D_{x,-it}u_1 \in \ker(D_{x,-it})$. Because $u \in L_p(\Omega)$ was arbitrary chosen our decomposition is a decomposition of the space $L_p(\Omega)$. □

In a similar way we can obtain a decomposition of the $L_p(\Omega)$ space in terms of the parabolic Dirac operator $D_{x,+it}$. Moreover, let us remark that the above decompositions are orthogonal in the case of $p = 2$.

Using the previous definitions we can also present an immediate application in the resolution of the linear Schrödinger problem with homogeneous boundary data.

Theorem 3.7. *Let $f \in L_p(\Omega)$, $1 < p \leq 2$. The solution of the problem*

$$\begin{cases} (-\Delta - i\partial_t)u &= f \ \ in \ \ \Omega \\ \\ u &= 0 \ \ on \ \ \partial\Omega \end{cases}$$

is given by $u = T_-Q_-T_-f$.

Proof. The proof of this theorem is based on the properties of the operator T_- and of the projector Q_-. Because T_- is the right inverse of $D_{x,-it}$, we get

$$D^2_{x,-it}u = D_{x,-it}(Q_-T_-f) = D_{x,-it}(T_-f) = f.$$

$\square$

4 The Non-Linear Schrödinger Problem

In this section we will construct an iterative method for the non-linear Schrödinger equation and we study is convergence. As usual, we consider the L_2-norm

$$||f||^2 = \int_\Omega [f\overline{f}]_0 dxdt,$$

where $[\cdot]_0$ denotes the scalar part.

Moreover, we also need the mixed Sobolev spaces $W_p^{\alpha,\beta}(\Omega)$. For this we introduce the convention

$$\begin{array}{rcl} \Omega^t = \{x : \ (x,t) \in \Omega\} & \subset & \mathbb{R}^m \\ \Omega^x = \{t : \ (x,t) \in \Omega\} & \subset & \mathbb{R}^+. \end{array}$$

Then, we say that

$$u \in W_p^{\alpha,\beta}(\Omega) \qquad \text{iff} \qquad \begin{cases} u(\cdot,t) \in W_p^{\alpha}(\Omega^t), & \forall\ t \\ \\ u(x,\cdot) \in W_p^{\beta}(\Omega^x), & \forall\ x \end{cases}$$

Under this conditions we will study the (generalized) non-linear Schrödinger problem:

$$\begin{array}{rcll} -\Delta_x u - i\partial_t u + |u|^2 u & = & f & \text{in } \ \Omega \\ u & = & 0 & \text{on } \ \partial\Omega, \end{array} \tag{11}$$

where $|u|^2 = \sum_A |u_A|^2$. We can rewrite (11) as

$$D^2_{x,-it}u + M(u) \quad = \quad 0, \tag{12}$$

where $M(u) = |u|^2u - f$. It is easy to see that

$$u \quad = \quad -T_-Q_-T_-(M(u)) \tag{13}$$

is a solution of (12) by means of direct application of $D^2_{x,-it}$ to both sides of the equation.

We remark that for $u \in W_2^{2,1}(\Omega)$, we get

$$||D_{x,-it}u|| = ||Q_-T_-M(u)|| = ||T_-M(u)||.$$

We now prove that (13) can be solved by the convergent iterative method

$$u_n \quad = \quad -T_-Q_-T_-(M(u_{n-1})). \tag{14}$$

For that purpose we need to establish some norm estimations. Initially, we have that

$$\begin{aligned} ||u_n - u_{n-1}|| &= ||T_-Q_-T_-[M(u_{n-1}) - M(u_{n-2})]|| \\ &\leq C_1||M(u_{n-1}) - M(u_{n-2})||, \end{aligned} \tag{15}$$

where $C_1 = ||T_-Q_-T_-|| = ||T_-||^2$.

We now estimate the factor $||M(u_{n-1}) - M(u_{n-2})||$. We get

$$\begin{gathered} ||M(u_{n-1}) - M(u_{n-2})|| = |||u_{n-1}|^2u_{n-1} - |u_{n-2}|^2u_{n-2}|| \\ \leq |||u_{n-1}|^2(u_{n-1} - u_{n-2})|| + |||u_{n-1} - u_{n-2}|^2u_{n-2}|| \\ \leq 2^{m+1}||u_{n-1} - u_{n-2}|| \left(||u_{n-1}||^2 + ||u_{n-2}||||u_{n-1} - u_{n-2}||\right), \end{gathered}$$

We assume $\mathcal{K}_n := 2^{m+1}\left(||u_{n-1}||^2 + ||u_{n-2}||||u_{n-1} - u_{n-2}||\right)$ so that

$$||u_n - u_{n-1}|| \leq C_1\mathcal{K}_n||u_{n-1} - u_{n-2}||.$$

Moreover, we have additionally that

$$\begin{aligned} ||u_n|| &= ||T_-Q_-T_-M(u_{n-1})|| \\ &\leq 2^{m+1}C_1||u_{n-1}||^3 + C_1||f|| \end{aligned} \tag{16}$$

holds.

In order to prove that indeed we have a contraction we need to study the auxiliary inequality

$$2^{m+1}C_1||u_{n-1}||^3 + C_1||f|| \leq ||u_{n-1}||,$$

that is,

$$||u_{n-1}||^3 - \frac{||u_{n-1}||}{2^{m+1}C_1} + \frac{||f||}{2^{m+1}} \quad \leq \quad 0. \tag{17}$$

The analysis of (17) will be made considering two cases

Case I: When $||u_{n-1}|| \geq 1$, we can establish the following inequality in relation to (17)

$$||u_{n-1}||^2 - \frac{||u_{n-1}||}{3 \cdot 2^{m+1}} + \frac{||f||}{2^{m+1}} \quad \leq \quad ||u_{n-1}||^3 - \frac{||u_{n-1}||}{2^{m+1}C_1} + \frac{||f||}{2^{m+1}}.$$

Then, from (17), we have

$$
\begin{aligned}
||u_{n-1}||^2 - \frac{||u_{n-1}||}{3\cdot 2^{m+1}} + \frac{||f||}{2^{m+1}} &\le 0 \\
||u_{n-1}||^2 - 2\frac{||u_{n-1}||}{6\cdot 2^{m+1}} + \frac{1}{36\cdot 2^{2m+2}} + \frac{||f||}{2^{m+1}} - \frac{1}{36\cdot 2^{2m+2}} &\le 0 \\
\left(||u_{n-1}|| - \frac{1}{6\cdot 2^{m+1}}\right)^2 \le \frac{1}{36\cdot 2^{2(m+1)}} - \frac{||f||}{2^{m+1}} & \\
= \frac{1}{2^{m+1}}\left(\frac{1}{36\cdot 2^{m+1}} - ||f||\right). & \qquad (18)
\end{aligned}
$$

If $||f|| \le \frac{1}{36\cdot 2^{m+1}}$ then

$$
\left| ||u_{n-1}|| - \frac{1}{6\cdot 2^{m+1}} \right| \quad \le \quad W,
$$

where $W = \sqrt{\frac{1}{36\cdot 2^{2(m+1)}} - \frac{||f||}{2^{m+1}}}$.

In consequence, if

$$
\frac{1}{6\cdot 2^{m+1}} - W \quad \le ||u_{n-1}|| \quad \le \frac{1}{6\cdot 2^{m+1}} + W
$$

then we have from (16) the desired inequality

$$
||u_n|| \le ||u_{n-1}||.
$$

Furthermore, we have now to study the remaining case. Assuming now that $||u_{n-1}|| \le \frac{1}{6\cdot 2^{m+1}} - W$, we have

$$
||u_n|| \le 2^{m+1} C_1 \left(\frac{1}{6\cdot 2^{m+1}} - W\right)^3 + C_1 ||f|| \le \frac{1}{6\cdot 2^{m+1}} - W
$$

and $||u_{n-1}|| \le \frac{1}{6\cdot 2^{m+1}} - W$, $||u_{n-2}|| \le \frac{1}{6\cdot 2^{m+1}} - W$ so that it holds

$$
||u_{n-1} - u_{n-2}|| \le 2\left(\frac{1}{6\cdot 2^{m+1}} - W\right).
$$

With the previous relations we can estimate the value of $\mathcal{K}_n$

$$
\begin{aligned}
\mathcal{K}_n &= 2^{m+1}\left(||u_{n-1}||^2 + ||u_{n-2}|| ||u_{n-1} - u_{n-2}||\right) \\
&\le 2^{m+1}\left[\left(\frac{1}{6\cdot 2^{m+1}} - W\right)^2 + 2\left(\frac{1}{6\cdot 2^{m+1}} - W\right)^2\right] \\
&\le 3\cdot 2^{m+1}\left(\frac{1}{6\cdot 2^{m+1}} - W\right) \\
&= \frac{1}{2} - 3\cdot 2^{m+1} W < \frac{1}{2}, \qquad (19)
\end{aligned}
$$

which implies that

$$||u_{n-2}|| \quad \leq \quad R := \frac{1}{3 \cdot 2^{m+1}}.$$

Finally, we have that

$$||u_n - u_{n-1}|| \quad \leq \quad \mathcal{K}_n ||u_{n-1} - u_{n-2}||,$$

with $\mathcal{K}_n < \frac{1}{2}$.

Case II: When $||u_{n-1}|| < 1$, we can establish the following inequality

$$||u_{n-1}||^4 - \frac{||u_{n-1}||^2}{3 \cdot 2^{m+1}} + \frac{||f||}{2^{m+1}} \quad \leq \quad ||u_{n-1}||^3 - \frac{||u_{n-1}||}{2^{m+1} C_1} + \frac{||f||}{2^{m+1}}.$$

Then, from (17), we have

$$\begin{aligned} ||u_{n-1}||^4 - \tfrac{||u_{n-1}||^2}{3 \cdot 2^{m+1}} + \tfrac{||f||}{2^{m+1}} \quad &\leq 0 \\ \Leftrightarrow \qquad \left(||u_{n-1}||^2 - \tfrac{1}{6 \cdot 2^{m+1}}\right)^2 \qquad &\leq \frac{1}{36 \cdot 2^{2m+2}} - \frac{||f||}{2^{m+1}}. \end{aligned} \tag{20}$$

Again, if $||f|| \leq \frac{1}{36 \cdot 2^{m+1}}$ then

$$\left| ||u_{n-1}||^2 - \frac{1}{6 \cdot 2^{m+1}} \right| \quad \leq \quad W,$$

where $W = \sqrt{\frac{1}{36 \cdot 2^{2m+2}} - \frac{||f||}{2^{m+1}}}$.

As a consequence,

$$\begin{aligned} \tfrac{1}{6 \cdot 2^{m+1}} - W \quad &\leq ||u_{n-1}||^2 \quad \leq \tfrac{1}{6 \cdot 2^{m+1}} + W \\ \Leftrightarrow \quad \sqrt{\tfrac{1}{6 \cdot 2^{m+1}} - W} \quad &\leq ||u_{n-1}|| \quad \leq \sqrt{\tfrac{1}{6 \cdot 2^{m+1}} + W} \end{aligned}$$

leads to $||u_n|| \leq ||u_{n-1}||$.

Again, considering now the case of $||u_{n-1}|| \leq \sqrt{\frac{1}{6 \cdot 2^{m+1}} - W}$, we obtain

$$||u_n|| \quad \leq 2^{m+1} C_1 \left(\sqrt{\tfrac{1}{6 \cdot 2^{m+1}} - W} \right)^3 + C_1 ||f|| \quad \leq \sqrt{\frac{1}{6 \cdot 2^{m+1}} - W}$$

and

$$||u_{n-1}|| \leq \sqrt{\tfrac{1}{6 \cdot 2^{m+1}} - W} \qquad ||u_{n-2}|| \leq \sqrt{\tfrac{1}{6 \cdot 2^{m+1}} - W} - W$$

$$||u_{n-1} - u_{n-2}|| \leq 2\sqrt{\tfrac{1}{6 \cdot 2^{m+1}} - W}.$$

With the previous relations we can estimate the value of $\mathcal{K}_n$

$$\begin{aligned}\mathcal{K}_n &= 2^{m+1}\left(||u_{n-1}||^2 + ||u_{n-2}||||u_{n-1}-u_{n-2}||\right)\\ &\leq 2^{m+1}\left[\left(\frac{1}{6\cdot 2^{m+1}} - W\right) + 2\left(\frac{1}{6\cdot 2^{m+1}} - W\right)\right]\\ &= 3\cdot 2^{m+1}\left(\frac{1}{6\cdot 2^{m+1}} - W\right)\\ &= \frac{1}{2} - 3\cdot 2^{m+1}W < \frac{1}{2}, \qquad (21)\end{aligned}$$

which implies that

$$||u_{n-2}|| \leq R := \frac{1}{3\cdot 2^{m+1}}.$$

Finally, we have that

$$||u_n - u_{n-1}|| \leq \mathcal{K}_n||u_{n-1} - u_{n-2}||,$$

with $\mathcal{K}_n < \frac{1}{2}$.

The application of Banach's fixed point, to the previous conclusions, results in the following theorem

Theorem 4.1. *The problem (11) has a unique solution* $u \in W_2^{2,1}(\Omega)$ *if* $f \in L_2(\Omega)$ *satisfies the condition*

$$||f|| \leq \frac{1}{36\cdot 2^{m+1}}.$$

Moreover, our iteration method (14) converges for each starting point $u_0 \in \overset{\circ}{W}{}_2^{1,1}(\Omega)$ *such that*

$$||u_0|| \leq \frac{1}{6\cdot 2^{m+1}} + W,$$

with $W = \sqrt{\frac{1}{36\cdot 2^{2(m+1)}} - \frac{||f||}{2^{m+1}}}$.

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